Handicap labelings of regular graphs

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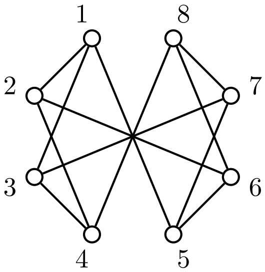
Handicap labeling

Definition

Let $G = (V, E)$ be a graph, $n = |V|$. The *handicap labeling* of G is a bijection $f : V \to \{1, 2, \ldots, n\}$ for which there exists an integer ℓ such that $\sum_{u \in N(v)} f(u) = \ell + f(v)v$ for all $v \in V$.



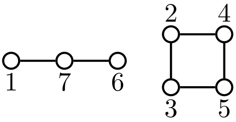
The sum is the *weight* $w_f(v)$ of v and all the weights in G form an arithmetic progression with a difference $d = 1$. Graph G is *handicap graph* if it admits a handicap labeling.

Related labelings

- ordered distance-antimagic vertex labeling (D. Fronček)
- distance magic labeling

Definition

Let $G = (V, E)$ be a graph, $n = |V|$. The *distance magic labeling* of G is a bijection $f : V \to \{1, 2, \ldots, n\}$ for which there exists an integer k such that $\sum_{u \in N(v)} f(u) = k$ for all $v \in V$.



The sum is the *weight* $w_f(u)$ of u and k is the *magic constant*. Graph G is *distance magic* if it admits a distance magic labeling.

some references

First introduced:

- M. Miller, C. Rodger and R. Simanjuntak: Distance magic labelings of graphs, *Australasian Journal of Combinatorics*, 28 (2003), 305–315.
- V. Vilfred: Σ-labelled graph and Circulant Graphs, *Ph.D. Thesis*, University of Kerala, Trivandrum, India, (1994).

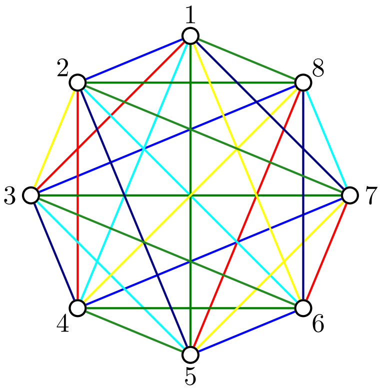
Today several papers, recent survey:

- S. Arumugam, et al.: Distance magic graphs — A Survey, *The Journal of Indonesian Mathematics Society*, Special Edition (2011), 11–26.

Tournament scheduling

Round robin tournament (RRT)

- n teams
- each team meets $n-1$ other teams (in $n-1$ weeks)

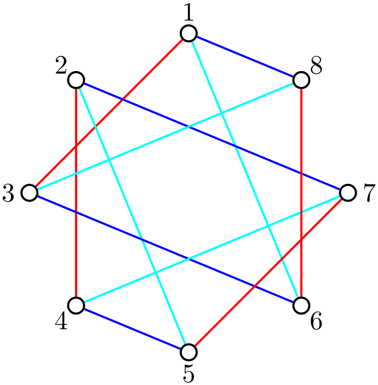


team	op. rankings
1	35
2	34
3	33
4	32
5	31
6	30
7	29
8	28

Incomplete tournament scheduling

Fair incomplete round robin tournament (FIT)

- n teams
- each team meets r opponents ($r < n - 1$)



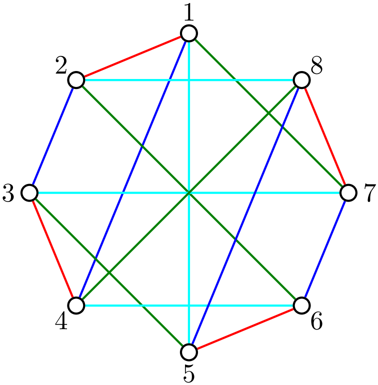
team	op. rankings	
1	35-k	17
2	34-k	16
3	33-k	15
4	32-k	14
5	31-k	13
6	30-k	12
7	29-k	11
8	28-k	10

- distance-antimagic vertex labeling

Incomplete tournament scheduling

Equalized incomplete round robin tournament (EIT)

- n teams
- each team meets r opponents $(r < n - 1)$



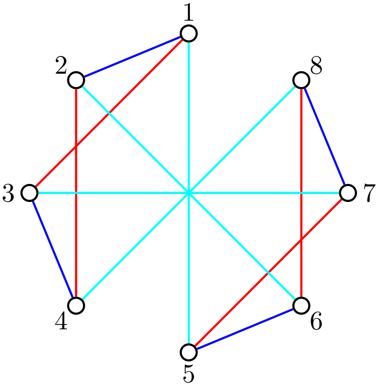
team	op. rankings	
1	k	18
2	k	18
3	k	18
4	k	18
5	k	18
6	k	18
7	k	18
8	k	18

- distance-magic vertex labeling

Incomplete tournament scheduling

Handicap incomplete round robin tournament (HIT)
(equal chance)

- n teams
- each team meets r opponents ($r < n-1$)



team	op. rankings	
1	$\ell + 1$	10
2	$\ell + 2$	11
3	$\ell + 3$	12
4	$\ell + 4$	13
5	$\ell + 5$	14
6	$\ell + 6$	15
7	$\ell + 7$	16
8	$\ell + 8$	17

- handicap labeling

Comparing incomplete tournaments

Fair incomplete tournament (FIT)

- each team meets the same number of opponents
- the total strength of opponents mimics the difficulty of a complete tournament

Equalized incomplete tournament (EIT)

- the total strength of opponents is the same
- precisely the definition of distance magic labeling

Observation

Regular graph G is EIT (distance magic) iff its complement $\overline{G}$ is FIT.

Handicap incomplete tournament (HIT)

- the strongest team has the strongest opponents

What are the requirements on HITs ?

Handicap incomplete tournament for n teams corresponds to a graph:

- r-regular with n vertices
- connected (does not have to be)
- has a handicap labeling

For which parameters n, r does a handicap graph exist?

Basic observations

The vertex weight

handicap graph G, n vertices, regularity r
the weight of every vertex is

$$\sum_{i=1}^{n}(\ell + i) = \sum_{i=1}^{n} w(i) = r\sum_{i=1}^{n} i$$

$$w(i) = \underbrace{(r-1)(n+1)/2}_{\ell} + i$$

The constant $\ell = (r-1)(n+1)/2$ for a regular handicap graph is determined uniquely.

Nonexistence

- not when n and r are both even (WE)

Further nonexistence

- no 1-regular handicap graph exists
- no 2-regular handicap graph exists
- no (n-1)-regular handicap graph exists
- no (n-2t)-regular handicap graph exists for any positive integer t (PP)
- not when $r \equiv 1 \pmod 4$ and $n \equiv 2 \pmod 4$ (POdd)

Overview of the cases r and n mod 4.

$n \backslash r$	$0 \pmod 4$	$1 \pmod 4$	$2 \pmod 4$	$3 \pmod 4$
$0 \pmod 4$	WE		WE	
$1 \pmod 4$		PP		PP
$2 \pmod 4$	WE	POdd	WE	
$3 \pmod 4$		PP		PP

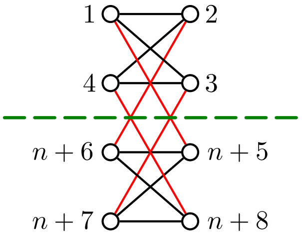
Theorem A

Let G be a 3-regular handicap graph with n vertices. Then a 3-regular handicap graph on $n+8$ vertices exists.

Sketch of the proof

Construction – by adding a component H G' with labeling f'

$$G' = G \cup H$$



$$f' : V(G') \to \{1, 2, \dots n+8\}$$

for $i \in V(H)$ and $i = 1, 2, 3, 4$ $j = i$
for $i \in V(H)$ and $i = 5, 6, 7, 8$ $j = i + n$
for $i \in V(G)$ $j = i + 4$
for $i \in V(H)$ $w(i) = 9 + i$
for $i \in V(G)$ $w(i) = n + 1 + i$

$$w'(j) = n + 9 + j$$

3-regular handicap graphs with n vertices

starting cases for inductive construction

- $n = 4$ $\nexists$ K_4 not Hendicap
- $n = 6$ $\nexists$ Comp $r = (n-3)$?
- $n = 8$ is Hendicap **Th. A**
- $n = 10$ $\nexists$ Comp
- $n = 12$ $\nexists$ Comp
- $n = 14$ $\nexists$ Comp
- $n = 16$ is Hendicap **Th. A**
- $n = 18$ $\nexists$ Comp
- $n = 20$ is Hendicap Comp
- $n = 22$ $\nexists$ Comp
- $n = 24$ is Hendicap **Th. A**
- $n = 26$ $\nexists$ Comp
- $n = 28$ is Hendicap **Th. A**
- $n = 30$ is Hendicap Comp
- $n = 32$ is Hendicap **Th. A**
- $n = 34$ is Hendicap Comp

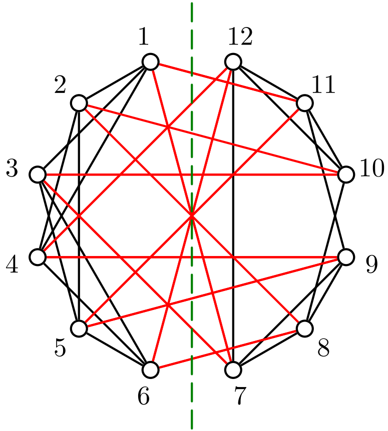
Construction by **Th. A** for $n = m + 8t$ where $m \geq 28$ is even and t is integer.

Theorem B (Kovář, K; 12)

Let G be a 5-regular handicap graph with n vertices. Then a 5-regular handicap graph on $n + 12$ vertices dexists.

Proofs idea

Construction – by adding a component H



$G' = G \cup H$

$f' : V(G') \to \{1, 2, \dots n + 12\}$

for $i \in V(H)$ and $i = 1, 2, \dots 6$ $\quad j = i$

for $i \in V(H)$ and $i = 7, 8, \dots 12$ $\quad j = i + n$

for $i \in V(G)$ $\quad j = i + 6$

for $i \in V(H)$ $\quad w(i) = 2 \cdot 13 + i$

for $i \in V(G)$ $\quad w(i) = 2(n + 1) + i$

$w'(j) = 2(n + 13) + j$

5-regular handicap graphs with n vertices

Starting cases for inductive construction

- $n = 6$ $\nexists$ $r = (n-1)$
  further not considering cases for $n \equiv 2 \pmod 4$ because of nonexistence **POdd**
- $n = 8$ $\nexists$ Comp $r = (n-3)$**?** A. Silber
- $n = 12$ is Hendicap **Th. B**
- $n = 16$ is Hendicap Comp
- $n = 20$ is Hendicap Comp
- $n = 24$ is Hendicap **Th. B**

Construction by **Th. B** for $n = m + 12t$, where t is integer, $m \geq 12$ and $m \equiv 0 \pmod 4$.

Overview of the cases r and n mod 4.

$n \backslash r$	0 (mod 4)	1 (mod 4)	2 (mod 4)	3 (mod 4)
0 (mod 4)	WE		WE	
1 (mod 4)		PP		PP
2 (mod 4)	WE	POdd	WE	
3 (mod 4)		PP		PP

* - we can get higher regularities using distance magic graphs

Thank you for your attention.