DiMaS

Využití spektrálních vlastností grafů pro ověření (ne)existence faktorizací

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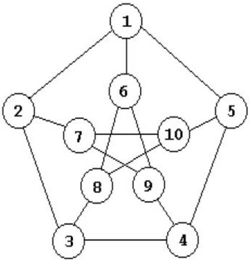
Graph factorization

"Can an input graph H be represented as an edge disjoint union of subgraphs $G_1, G_2, \ldots, G_k$, all of which are isomorphic to G?"

Petersen graph



(a) Petersen graph

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

(b) Its adjacency matrix

Question: **Can be the complete graph on 10 vertices decomposed into 3 copies of the Petersen graph?**

Analysis

$$det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4$$

1. The corresponding eigenvector to the largest eigenvalue $\lambda_{max} = 3$ is the constant vector $v = (1, 1, \dots, 1)$ (further referred as **1**).
2. We are looking for the decomposition of complete graph on 10 vertices decomposed into 3 copies of the Petersen graph.

   $$\mathbf{A} = J - I = A_1 + A_2 + A_3$$

   where **A** denotes the adjacency matrix of complete graph and $A_i = PAP^T = PAP^{-1}$.
3. Also, A_i is a symmetric matrix and hence the dimension of an eigenspace of eigenvalue λ (*geometric multiplicity*) is the multiplicity of λ as a root in the characteristic equation for A_i (*algebraic multiplicity*).
4. Remark, that $det(\mathbf{A} - \lambda I) = (\lambda - 9)(\lambda + 1)^9$ and **1** is corresponding eigenvector to the eigenvalue 9.

Analysis

$$det(A - \lambda I) = (\lambda - 3)(\lambda - 1)^5(\lambda + 2)^4$$

5. $\mathbf{1}$ is an eigenvector of eigenvalue $\lambda_1 = 3$ of all three A_i and so each has a 5-dimensional eigenspace of eigenvalue $\lambda_2 = 1$ orthogonal to $\mathbf{1}$ and hence contained in $(span(\mathbf{1}))^\perp$ which is a 9-dimensional space.
6. **Any two** 5-dimensional subspaces of a 9-dimensional space must have a non zero vector, say v, in their intersection. Thus there is a vector v that is an eigenvector of eigenvalue $\lambda_2 = 1$ for both A_1 and A_2 ($A_1 \cdot v = A_2 \cdot v = 1 \cdot v$) and moreover v is orthogonal to $\mathbf{1}$ ($\mathbf{1} \cdot v = 0$).

$$\begin{aligned}
(J - I)v &= Jv - v = 0 - v = -v, \\
J - I &= A_1 + A_2 + A_3, \\
(J - I)v &= (A_1 + A_2 + A_3)v = A_1 v + A_2 v + A_3 v, \\
-v &= 2v + A_3 v, \\
A_3 v &= -3v.
\end{aligned}$$

This is a contradiction since -3 is not an eigenvalue of A_3.

Generalization

Let us consider decomposition to two graphs (for simplification):

$$\mathbf{A} = A_1 + A_2, \quad A_2 = PA_1P^T.$$

What about spectral decomposition?

$$\begin{aligned} V^T D_A V &= U_1^T D_1 U_1 + U_2^T D_2 U_2, \\ V^T D_A V &= U_1^T D_1 U_1 + P^T U_1^T D_1 U_1 P, \\ V^T D_A V &= U_1^T D_1 U_1 + (U_1 P)^T D_1 (P^T U_1^T)^T, \end{aligned}$$

A_1, A_2 have the same eigenvalues, but corresponding eigenvectors of A_2 are permuted ($v_{A_2} = P \cdot v_{A_1}$).

$$Av = A_1 v + PA_1 P^T v.$$

Extension by Padraic Barlett (?)

We might not know the precise eigenvalues - but rather just (say) their signs: one of them will have lots of negative small eigenvectors, while the other will just have one or two rather big negative ones ... Lagrangian

$$f_G(v) = \langle Av, v \rangle = \langle Av \rangle^T v = \sum_{(i,j) \in E(G)} v_i v_j .$$

Proposition

Given a graph G, let W^+ denote the space generated by all of the positive eigenvectors of A_G, W^- the space generated by all of the negative eigenvectors, and W^0 the eigenspace corresponding to 0. Notice that because A_G is symmetric, by the spectral theorem, we can write any element in $\mathbb{R}^n$ as a sum of one element from each of these spaces. We claim that our function f_G is positive-semidefinite on the space $W^+ \oplus W^0$ (i.e. the space generated by all of the nonnegative eigenvectors,) and negative-semidefinite on the space $W^- \oplus W^0$.

Theorem

The complete graph K_n cannot be decomposed into less then $n-2$ complete bipartite graphs.

Lemma 8.5.1
Let X be a k-regular graph on n vertices with eigenvalues $k, \theta_2, \ldots, \theta_n$. Then X and its complement $\overline{X}$ have the same eigenvectors, and the eigenvalues of $\overline{X}$ are $n - k - 1, -1 - \theta_2, \ldots, -1 - \theta_n$.

Sketch of proof: *(p.172)*

$$A(\overline{X}) = J - I - A(X),$$

$$A(\overline{X})u_i = (J - I - A(X))u_i = (-1 - \theta_i)u_i, \quad u_i \perp \mathbf{1}.$$

Thank you for your attention.