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SQUARE OF METRICALLY REGULAR GRAPHS

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Let X be a finite set, $n := |X| \geq 2$. For an arbitrary natural number D let $\mathbf{R}= \{R_0, R_1, \ldots, R_D\}$ be a system of binary relations on X. A pair $(X,\mathbf{R})$ will be called *an association scheme* with n classes if and only if it satisfies the axioms $A1 - A4$:

A1. The system $\mathbf{R}$ forms a partition of the set X^2 and R_0 is the diagonal relation, i.e. $R_0 = \{(x, x); x \in X\}$.

A2. For each $i \in \{0, 1, \ldots, D\}$ it holds $R_i^{-1} \in \mathbf{R}$.

A3. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds
$(x, y) \in R_k \wedge (x_1, y_1) \in R_k \Rightarrow p_{ij}(x, y) = p_{ij}(x_1, y_1)$, where $p_{ij}(x, y) = |\{z; (x, z) \in R_i \wedge (z, y) \in R_j\}|$.
Then define $p^k_{ij} := p_{ij}(x, y)$ where $(x, y) \in R_k$.

A4. For each $i, j, k \in \{0, 1, \ldots, D\}$ it holds $p^k_{ij} = p^k_{ji}$.

The set X will be called the *carrier* of the association scheme $(X,\mathbf{R})$. Especially, $p^k_{i0} = \delta_{ik}$, $p^0_{ij} = v_i\delta_{ij}$, where δ_{ij} is the Kronecker-Symbol and $v_i := p^0_{ii}$, and define $P_j := (p^k_{ij})$, $0 \leq i, j, k \leq D$.
Given an undirected graph $G = (X, E)$ of diameter D we may now define $R_k = \{(x, y); d(x, y) = k\}$, where $d(x, y)$ is the distance from the vertex x to the vertex y in the standard graph metric. If $(X,\mathbf{R})$ gives rise to an association scheme, the graph G is called *metrically regular* (sometimes also called *distance regular*) and p^k_{ij} are said to be its *parameters*. In particular, a metrically regular graph with diameter $D = 2$ is called *strongly regular*.
Let $G = (X, E)$ be an undirected graph without loops and multiple edges. *The second power* (or *square* of G) is the graph $G^2 = (X, E')$ with the same vertex set X and in which mutually different vertices are adjacent if and only if there is at least one path of the length 1 or 2 in G between them.
The necessary conditions for G to have the square G^2 metrically regular are found and some constructions of those graphs are solved.