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CRITICALITY IN SPERNER'S LEMMA

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Sperner's lemma states that if a labelling of the vertices of a triangulation K of the d-simplex Δ^d with labels $1, 2, \ldots, d+1$ has the property that (i) each vertex of Δ^d receives a distinct label, and (ii) any vertex lying in a face of Δ^d has the same label as one of the vertices of that face, then there exists a rainbow facet (a facet whose vertices have pairwise distinct labels).

Tibor Gallai asked in 1969 whether Sperner's Lemma is 'critical' in the sense that for every triangulation K as above and every facet σ of K, there is a labelling satisfying (i) and (ii) such that σ is the unique rainbow facet. (The question is included as Problem 9.14 in Jensen and Toft's collection *Graph Coloring Problems*.)

In this talk, we show that the answer is affirmative for $d \leq 2$ (as already proved by Gallai). For every $d \geq 3$, however, we answer Gallai's question in the negative by constructing an infinite family of examples where no labelling with the requested property exists. The construction is based on the properties of a convex 4-polytope which had been used earlier to disprove a claim of Theodore Motzkin on neighbourly polytopes.

Joint work with Matěj Stehlík and Riste Škrekovski.